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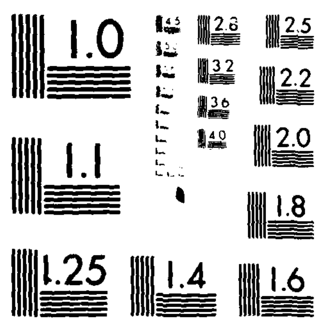
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COHERENCE THROUGH PARTIAL INFORMATION IN AN ADDITIVE MULTIATTRIBUTE UTILITY ANALYSIS

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SUMMARY

This report addresses the specific problem of the resolution of incoherent weight assessments in an additive multiattribute utility analysis. The approach taken is that if incoherencies occur because actual numerical assessments are too precise, then it would be useful if the ramifications of less precise, but coherent, information were made clear. Two types of information are considered. First, it is supposed that the decision maker can order the attributes on the relative importance of the weights. The implications of any given ordering are shown to be very simply analyzed. Second, it is supposed that, in addition, inequality assessments can be made between certain pairs of weights. The analysis demonstrates the implications of these, and also suggests which inequality assessments are likely to be most useful.

1.0 INTRODUCTION

Much attention has recently been paid to the problem that arises in practical decision analysis when a decision maker fails to comply with the axiomatic system on which the theory of decision analysis is based. Lindley, Tversky, and Brown (1979) coined the term "incoherence" to describe this phenomenon. Part of the unease about incoherence seems to have been brought about by the fact that many decision-analytic assessments are necessarily subjective, and so difficult, if not impossible, to verify. This means that we tend to believe that people will be incoherent whether the property has been observed or not.

Previous work in this area has been almost exclusively concerned with probability judgments. There have been three approaches taken in dealing with incoherence, which are based on one's degree of conviction about the axiomatic system that underlies the assessments. At one extreme, the axiomatic system is assumed to be correct, but the "measurements" are inconsistent. The problem then becomes one of trying to use the observed measurements to provide a best guess at the "true" measurement (Lindley et al.,1979; Freeling, 1981a). At the other extreme, the axiomatic system is rejected for some reason, and attempts are made to produce more appropriate systems. Some of these approaches have come under the heading: "extended theories of belief" (Smith, 1965; Dempster, 1968; Shafer, 1976; Freeling, 1980). Between these extremes lies the approach that accepts the axiomatic system as an ideal system, but one which in practice is unattainable. The theory states that a solution exists, but no algorithm can be found that will produce the solution. In this case, it is useful to see if the information one has can produce any insights into the problem, and perhaps produce a useful decision aid. Such approaches often use the phrase "partial information" to describe them (Potter and Anderson, 1980).

In this paper, we consider how to tackle the incoherencies that arise in multiattribute utility analyses. Furthermore, the consideration is limited to additive models rather than multiplicative models. Multiattribute utility analysis is an increasingly popular tool for decision analysts, and more often than not additive models are assumed (e.g., Edwards, 1977). (A recently published example is by Snapper and Seaver (1980).) Two sets of numerical assessments are required in such an analysis. For each attribute defined, the options under consideration must be scored (which scores are sometimes objective and sometimes subjective), and then weights which are typically subjective must be assessed for each attribute. Incoherencies can arise in both sets of assessments, but this paper only considers incoherencies in the weights, firstly because they are generally the more difficult and subjective assessments to make, and secondly because incoherencies in weights are more readily determined.

We assume then that an additive multiattribute utility analysis of a given problem has been tackled, that a set of attributes and a set of options have been identified, and that the options have been satisfactorily scored on each attribute. We are left to try and assess weights across each attribute. This is usually done via a pairwise comparison between attributes. Obvious incoherencies arise when for three attributes, i, j, and k, comparisons between i and j and between j and k do not correspond with the comparison between i and k, which can be directly computed from the other two comparisons. Practitioners will usually perform some sensitivity analyses on the weights, but computerized sensitivity analyses tend to be rather simple, and manual ones very laborious.

In examining ways to approach the incoherencies that arise in this sort of analysis, we could take any of the three paths that were outlined above. The axiomatic system for multiattribute decision analysis has been criticized (e.g., Rivett, 1972), but it is fair to say that although alternative procedures have been suggested, compelling alternative axioms have not been forthcoming. The arguments rather complain that the axiom system is too demanding.

Instead, then, we look at the possibility of accepting the decision-analytic axioms as an ideal system, but assume that the decision maker is not able to meet the demands that the system imposes. An approach that takes a set of incoherent weight assessments and provides a set of coherent ones has been suggested by Freeling (1981a). The problem with such an approach is that it can be very difficult to explain to the decision maker exactly why a certain set of assessments, that were not directly obtained, should be used instead of a directly obtained set. This is a serious problem, because the purpose of a decision analysis is, in our view, to provide the decision maker with an understanding of the problem and the issues that are at stake in it, rather than to produce an optimal decision, as does an optimization routine. We therefore consider looking at the kind of partial information that a decision maker might both feel confident about and be coherent about; and see what this information might have to say about the decision, and how it might be easily presented.

In order correctly to link the weight assessments with the scores, weights are usually obtained by pairwise comparison between attributes that have been scored. If we accept that such assessments of weights are (at least potentially) incoherent, what information might we expect to be coherent? A very simple statement about two attributes is that one should be weighted higher than another. This is an ordinal rather than a cardinal ranking, and is one which, we believe, can be made quite confidently in such analyses. The first assumption we shall make, then, is that the decision maker is able to rank the attributes in order of weight size. It will also become apparent that the analysis will help considerably in studying the problem where the relative positions of two attributes are not known with complete confidence.

In addition to this information, it is likely that the decision maker will be able to make certain quantitative statements about the pairwise assessments. In a similar approach to that adopted in the subjective probability assessment literature (see Freeling, 1981b, for a discussion of these), we assume that although a point assessment will not be coherent, the subject can place upper and lower bounds on the assessment. However, we shall not

insist that this can be done for all pairs of assessments. Rather, our investigations will be to see what can done with given assessments, and to suggest which assessments are most important.

2.0 ANALYSIS

In this section, we consider how one might incorporate in an analysis two sorts of information that might be forthcoming. The additive multi-attribute problem is:

$$\max_j \sum_i w_i u_{ij}$$

where w_i is the weight on attribute i, and u_{ij} is the score of option j on attribute i, assumed to be known explicitly. Note that if the w_i were known, we could simply perform the maximization. First, we demonstrate how one can very simply incorporate a preference ordering over the weights. Second, we see how the introduction of further inequality constraints, of the form $w_\ell \leq$ or $\geq \lambda w_m$ can also be incorporated quite simply. With the mathematical formulation developed in this section, practical considerations using an example will be highlighted in the next section.

2.1 The Effect of an Ordering on the Weights

Suppose that the decision maker is able to provide, in addition to the inputs already mentioned, a consistent ordering over the weights w_i. This means that, by renumbering if necessary, we can say that $w_1 \geq w_2 \geq \cdots \geq w_n$. Ideally, we might hope that this information would indicate that a particular option must be best, but failing that, it could be very useful if it resulted in one option being clearly unfavorable. In a choice between two options, the problem of discovering whether one option is preferred to the other whatever the weights are, supposing only that they satisfy the inequality constraints above, can be illuminated by considering the following linear program:

$$\text{Maximize} \quad \sum_{i=1}^{n} w_i (u_{ik} - u_{ij})$$

subject to:

$$w_1 > w_2$$
$$\vdots$$
$$w_{n-1} > w_n$$

and

$$\Sigma w_i = 1.$$

If this maximum is less than zero, then option j must be preferred to k for any allowable weights. Similarly, if the minimum of the same objective function were always greater than zero, then k must be preferred to j for any allowable weights. In other cases, we cannot make any preference statement between the two options, and this presents a problem. This analysis does not indicate what region in the allowable space will produce positive or negative scores on the objective function. The problem is complicated further if there are more than two options. In this case, a given option (option 1 say) cannot be best if some convex combination of the other options is better than it for any allowable set of weights. For in this case, we know that for any allowable set of weights there is always one option that is better than option 1.

Mathematically, we wish to see if there exists some x_j, $j=2,\ldots,m$, where $x_j > \emptyset$ for all j, and $\sum_j x_j=1$, such that:

$$\sum_{i=1}^{n} w_i(u_{i1} - \sum_{2}^{m} x_j u_{ij})$$

is less than zero for any allowable set of weights w_i. This is a much less simple linear program.

It would be much more illuminating if we could find a procedure which made the question of dominance little more than a matter of inspection. We can do this by reformulating the problem is such a way that the ordering constraints on the weights effectively can be hidden in the formulation, thus allowing for simple intuitive analysis of the problem. We can "hide" the ordering conditions by rewriting the formula $\Sigma w_i u_i$ (suppressing the option subscript j for convenience) as follows:

$$\Sigma w_i u_i = nw_n \sum_1^n \frac{u_i}{n} + \ldots + j(w_j - w_{j+1}) \sum_1^j \frac{u_i}{j} + \ldots + (w_1 - w_2)u_1$$

Now, define α_i and v_i as follows:

$$\alpha_i = i(w_i - w_{i+1}) \quad \text{for } i<n$$

$$\alpha_n = nw_n$$

and

$$v_i = \frac{1}{i} \sum_1^i u_j$$

and we have that:

$$\Sigma w_i u_i = \Sigma \alpha_i v_i$$

Furthermore, it is easily verified that the α_i are true weights, in that they are all positive, between Ø and 1, and they must sum to 1. Moreover, each α_i can attain its maximum of 1 and minimum of Ø. The latter will be true if $w_i = w_{i+1}$. To prove that the maximum is 1, and can be attained, note that if $j(w_j - w_{j+1}) > 1$, then $w_j > w_{j+1} + 1/j$. But because $w_1 \geq w_2 \ldots \geq w_j$, the maximum value of w_j is $1/j$, as $\sum_{i=1}^{j} w_i \leq 1$, which, with $w_{j+1} \geq 0$, provides a contradiction. By the same token, $j(w_j - w_{j+1}) = 1$ holds only if $w_{j+1} = Ø$. It can be further verified that the equality <u>will</u> hold if $w_i = 1/j$ for i = 1 to j, and $w_i = Ø$ for $i > j$. Note also that the new "score," v_j is the average of the first j old scores.

The converse expressions for w_i and u_i are:

$$w_i = \sum_i^n \alpha_j / j$$

and

$$u_i = iv_i - (i-1)v_{i-1} \quad \text{for } i>1,$$

$$u_1 = v_1.$$

We can also demonstrate that an option can never be best, if and only if that option is strictly dominated by some convex combination of the other

options in $\underline{\alpha}$ $\underline{v}$ space. First, suppose option 1 is strictly dominated by a convex combination of the other options. This means that there exist x_j, $j=2,\ldots,m$, $x_j \geq \emptyset$ and $\Sigma x_i=1$ such that:

$$v_{i1} < \sum_{2}^{m} x_j v_{ij} \text{ for all } i,\ i=1,\ldots,n. \tag{1}$$

Now suppose, that there exists a set of α_i, $i=1,\ldots n$ such that option 1 is preferred to all other options, i.e.,

$$\sum_{1}^{n} \alpha_i v_{i1} > \sum_{1}^{n} \alpha_i v_{ij} \text{ for all } j,\ j=2,\ldots,m.$$

Multiplying each of these inequalities by x_j, and summing, we obtain that:

$$\sum_{2}^{m} x_j \sum_{1}^{n} \alpha_i v_{i1} > \sum_{2}^{m} x_j \sum_{1}^{n} \alpha_i v_{ij},$$

or, on rearranging, that, (as $\sum_{2}^{m} x_j=1$),

$$\sum_{1}^{n} \alpha_i v_{i1} > \sum_{1}^{n} \alpha_i \sum_{2}^{m} x_j v_{ij},$$

so that (from (1)):

$$\sum_{1}^{n} \alpha_i v_{i1} > \sum_{1}^{n} \alpha_i v_{i1}$$

which is a contradiction.

Conversely, suppose that option 1 is not dominated, so that there exist no x_j that satisfy (1). Consider the space of attributes. Let y be the point that represents option 1. Let X be the set of points that represents all convex combinations of the remaining n-1 points. Then y does not belong to X. Let Z be the union of X with all points dominated by any point in X. Then y does not belong to Z, and Z is convex. This implies that there is a supporting hyperplane defined by the equation $\beta^T x = c$ that divides the point y from the set Z, i.e., for which $\beta^T y > c$ and $\beta^T z \leq c$ for all z in Z. Hence, $\beta^T y > \beta^T z$ for all z in Z. From our

choice of Z, we can choose z_i sufficiently large negative so that if β_i were negative, $\beta^T z$ could be arbitrarily large, and in particular, larger than $\beta^T y$. Hence, each β_i in β must be non-negative. Let α_i be the normalized equivalent of β_i. Then we have proved that:

$$\Sigma\alpha_i y_i > \Sigma\alpha_i z_i$$

for all z in Z, and in particular for all z in X. That is, for this set of α_i, option 1 is better than any combination of all other options.

Therefore, in order to see what information about the relative merits of different options is provided just by information about the order of the weights, we can perform the transformation described above, to see if any of the options are now dominated. If so, then those options can be ruled out. If not, then more information about the weights is required before any options definitely can be ruled out.

2.2 Inequality Comparisons Between Attributes

Suppose now that the decision maker has been able to provide an ordering over the attribute weights, but that there are still at least two options which are not dominated. We can now demonstrate how we can incorporate inequality comparisons between the attributes to see if this will result in any options being discarded.

2.2.1 A single assessment.

First, consider just one assessment. Suppose that for attributes k and ℓ, where $k<\ell$, i.e., $w_\ell < w_k$, we have the information that $w_\ell \leq \lambda_1 w_k$. In terms of the α_i, this means that

$$\sum_{k}^{n}\alpha_i/i \leq \lambda_1\sum_{\ell}^{n}\alpha_i/i$$

At first sight it is not clear what this condition implies for the α_i. However, let us reconsider what we hope to achieve by obtaining this information. We hope eventually to show that one option is to be preferred to all others. It would be useful, then, if the imposition of

this constraint made it possible to rule out one of the contending options. In ruling out one of the contending options, what we have to show is that the maximum of the score of this option is always less than the equivalent score of some convex combination of the alternatives, given the constraint as was shown in Section 2.1. Algebraically, we hope to show that for a given option which we may call option 1, there exists some convex combination of alternatives such that:

$$\max_{\alpha_i} \sum_{i=1}^{n} \alpha_i (v_{1i} - \sum_{2}^{m} x_j v_{ji}) < \emptyset$$

subject to the constaints:

$$\sum_{k}^{\ell-1} \alpha_i / i \leq (\lambda_1 - 1) \sum_{\ell}^{n} \alpha_i / i$$

and

$$\Sigma\alpha_i = 1 \qquad (\alpha_i \geq \emptyset \text{ for all } i).$$

For if this can be shown, then whatever the weights α_i are, option 1 can never be best.

This is a linear program, and we can use many of the standard results of linear programming to analyze this problem. In particular, we can observe that, because there are only two constraints, a basic solution will have at most two non-zero α_i. There are three possible forms to the solution of this problem. Let the maximum without the constraint be when α_x=1 for some x. Then either α_x=1 satisfies the constraint, in which case the maximum does not change, or the constraint holds, i.e., the inequality can be replaced by an equality constraint (by complementary slackness). If the constraint holds, then either all the α_i in this constraint are zero, in which case there exists an i less than k such that α_i=1, or there exist i and j, where $k \leq i < \ell \leq j$, such that the constraint holds. This means that:

1) $\frac{\alpha_i}{i} = \frac{(\lambda_1-1)\alpha_j}{j}$, and

2) $\alpha_i + \alpha_j = 1.$

The first expression gives:

$$\alpha_i = \frac{(\lambda_1-1)i\alpha_j}{j}$$

substituting 2):

$$\rightarrow \quad j\alpha_i = i(\lambda_1-1)(1-\alpha_i)$$

$$\rightarrow \quad (j+i(\lambda_1-1))\alpha_i = i(\lambda_1-1)$$

$$\rightarrow \quad \alpha_i = \frac{i(\lambda_1-1)}{i(\lambda_1-1)+j}.$$

Similarly

$$\alpha_j = \frac{j}{i(\lambda_1-1)+j}.$$

The maximum in this case will be:

$$\frac{(\lambda_1-1)i\Delta_i+j\Delta_j}{i(\lambda_1-1)+j}$$

where

$$\Delta_i=v_{1i}-\sum_2^m x_jv_{ji}.$$

If this maximum is less than zero, then we know that option 1 cannot be optimal. Note, however, that if this maximum is greater than zero, then its exact value is of little interest. This prompts a simplification of this formula for ease of analysis. For, if instead of finding the pair of i and j to optimize this equation, we find those that optimize the

numerator, and because the denominator is always greater than zero, it follows that the optimum of the numerator is less than or greater than zero if and only if the optimum of the whole is less than or greater than zero, although the optimal i and j in each case may be different. But the optimum of the numerator is very simple to find: all we have to do is find the i such that $k \leq i < \ell$ and $i\Delta_i$ is maximized, and a similar j such that $j \geq \ell$ and $j\Delta_j$ is maximized. There is still the problem of assessing what the x_j should be, especially as the use of the constraint will depend on the x_j, but if, as is most simple, the x_j were set, particularly if x_k was set to 1 for some k, the analysis would become quite simple.

2.2.2 The general case. Consider now the case where there are several such constraints. For a given objective function, we know from ideas of complementary slackness in linear programming that a basic solution has exactly one non-zero variable for each constraint that is met. Consider then just the constraints that are met. Such a constraint is of the form:

$$\sum_{x}^{n}\frac{\alpha_i}{i} = \lambda_{xy}\sum_{y}^{n}\frac{\alpha_i}{i}$$

Let α_{ix} be the first non-zero α_i for i greater than or equal to x. Then, in terms of the non-zero α_i, this is equivalent to:

$$\frac{\alpha_{ix}}{i_x} + \frac{\alpha_{ix+1}}{i_{x+1}} + \ldots + \frac{\alpha_{ie}}{i_e} = \lambda_{xy}\left(\frac{\alpha_{iy}}{i_y} + \ldots + \frac{\alpha_{ie}}{i_e}\right)$$

where i_e is the last non-zero α_i. Now, there is one more non-zero variable than this type of contraint, the summation to unity being the extra constraint. This implies two things. First, if we consider just the constraints except the summation to unity, there can only be one solution for each α_i in terms of one non-zero α_j. Second, because of the type of constraint that exists, and the fact that they are now equality constraints, we can find constants which are equivalent to λ_{xy}, where y=x+1, and are either directly obtained from a constraint, or can be calculated from sets of constraints, and are the ratios between w_x and w_{x+1}. Define $\lambda_{x(x+1)}$ to be λ_x. Then, λ_{xy} can be rewritten as $\lambda_x\lambda_{x+1}\ldots\lambda_{y-1}$.

We are now in a position to verify the following result:

Theorem

The solution to this set of linear equations has a solution for α_{i_x} of the form:

$$\frac{\alpha_{i_x}}{i_x} = \Lambda_x \frac{\alpha_{i_e}}{i_e}$$

where $\Lambda_x = (\lambda_x - 1)\lambda_{x+1} \cdots \lambda_{e-1}$.

Proof

It will be verified that this form is indeed a solution to the set of equations. Remembering that the general equation is of the form:

$$\sum_x^n \frac{\alpha_i}{i} = \lambda_{xy} \sum_y^n \frac{\alpha_i}{i}$$

and that $\lambda_{xy} = \lambda_x \lambda_{x+1} \cdots \lambda_{y-1}$, the theorem will hold if we can show that:

$$\sum_x^n \frac{\alpha_i}{i} = \prod_x^e \lambda_i .$$

Consider the first two terms of the left hand side of this equation, under the hypothesis. The multiplier of the term is $\Lambda_x + \Lambda_{x+1}$, and

$$\Lambda_x + \Lambda_{x+1} = \{(\lambda_x - 1)\lambda_{x+1} + (\lambda_{x+1} - 1)\} \prod_{x+2} \lambda_i$$

$$= (\lambda_x \lambda_{x+1} - 1) \prod_{x+2} \lambda_i .$$

Now, if we add the next term, we can see that in like fashion λ_{x+2} will be transferred from the latter product to the former. Finally, by observing that $\Lambda_e = 1$, we can see that in the end the -1 term in these equations will disappear, leaving us with just the former product, i.e.,

$\prod_{x}\lambda_i$, as we were trying to prove. Thus the hypothesis is verified.

This result is very useful, because it means that it is quite easy to analyze good basic solutions. For example, suppose we were interested in seeing if the constraints implied that one option was worse than another, so that the objective function is of the form:

$$\max_{\alpha_i} \sum_{i=1}^{n} \alpha_i (v_{1i}-v_{2i}).$$

Letting $\Delta_i=v_{1i}-v_{2i}$, the actual value of the objective funciton for this set of basic solutions is:

$$\frac{\sum_{1}^{e} \Lambda_{j_i} j_i \Delta_{j_i}}{\Sigma \Lambda_{j_i} j_i}$$

Again, because we are only interested in whether or not this objective function can become greater than zero, all we need do is look at the numerator of this expression. Furthermore, it becomes quite simple to see what effect the addition of a new constraint will have on the problem.

This provides the mathematical underpinings to the problem of analyzing the two sorts of information that we have suggested to be worth considering in the context of weight assessment. Considerations of practical interest are discussed in the next section.

3.0 AN EXAMPLE

In order to see why the above analysis can be of help to decision analysts, we consider a practical example*.

The scores that are assumed are presented in Table 1. Each column represents an option, and each row an attribute.

TABLE 1

X_i \ O_j	1	2	3	4	5	6	7	8	9
1	.715	.800	.875	.760	.780	.885	.740	.945	.935
2	.990	.990	.990	.998	.994	.998	.999	.991	.992
3	.791	.791	.782	.622	.457	.457	.726	.778	.898
4	.980	.980	.860	.880	.760	.980	1.000	1.000	1.000
5	.726	.707	.707	.575	.675	.546	.558	.509	.659
6	.958	1.000	.968	.960	.730	.603	.898	.762	.751

Following the original example, the ordering over the weights is assumed to be:

$$w_6 > w_1 > w_2 > w_5 > w_4 > w_3 {}^{**}.$$

Note first that any dominance relations in Table 1 are not immediately apparent. However, by making the transformation of Section 2.1 we obtain the new table:

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*The example is based on the Nuclear Power Siting Problem of Keeney (1980), which although not an additive model, provides an excellent example. The scores presented in Table 1 were gleaned from the text of this book.

**N.B. that this was the first information to be obtained in the study.

TABLE 2

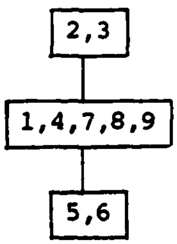
α_i \ O_j	1	2	3	4	5	6	7	8	9
1	95.8	100	96.8	96.0	73.0	60.3	89.8	76.2	75.1
2	83.7	90.0	92.2	86.0	75.5	74.4	81.9	85.4	84.3
3	88.8	93.0	94.4	90.6	83.5	82.9	87.9	89.9	89.3
4	84.7	87.4	88.5	82.3	79.5	75.8	79.9	80.2	83.4
5	87.4	89.5	88.0	83.5	78.8	80.2	83.9	84.4	86.7
6	86.0	87.8	86.4	79.9	73.3	74.5	82.0	83.0	87.3

Here the scores have been multiplied by one hundred for convenience. Table 2 yields a number of obvious dominance relationships. They can be categorized as in Table 3:

TABLE 3

option	dominates	is dominated by
1	5,6,7	2,3
2	1,4,5,6,7,8,9	-
3	1,4,5,6,7,8	-
4	5,6	2,3
5	-	1,2,3,4,7,8,9
6	-	1,2,3,4,7,8,9
7	5,6	1,2,3
8	5,6	2,3
9	5,6	2

This immediately suggests the following preference structure:



and if we are looking for only one option, then it must be either option 2

or option 3. Thus, we can see very easily exactly what implications can be drawn from the information about the preference ordering over the weights. In this case, quite a staggering amount of information is obtained, and we believe that this is likely often to be the case when the number of options is similar to the number of attributes, or larger.

However, this analysis does not give us the complete story, and so we can turn to the second part of the analysis to see what further information will help us to decide between option 2 and option 3. Let Δ_i be the difference between the score on option 2 and that on option 3. The information we shall require is listed in Table 4:

TABLE 4

α-attribute	w-attribute	option 2	option 3	Δ_i	$i\Delta_i$
α_1	w_6	100	96.8	3.2	3.2
α_2	w_1	90.0	92.2	-2.2	-4.4
α_3	w_2	93.0	94.4	-1.4	-4.2
α_4	w_5	87.4	88.5	-1.1	-4.4
α_5	w_4	89.5	88.0	1.5	7.5
α_6	w_3	87.8	86.4	1.4	8.4

Note that because some of the Δ_i are positive and some are negative, it is possible to choose combinations of the α-attributes so that either option 2 is better or option 3 is better. In order to be able to distinguish between these options, more information is required, and so the imposition of an inequality constraint is now considered.

In what follows, we do not pretend to perform an exhaustive analysis of the use of inequality constraints. Rather we shall use the example to illuminate some useful, and hopefully typical, aspects of this type of information. However, it is likely that the best approach will vary from problem to problem. Returning to the example, then, we can see that a comparison between w_6 and w_1 looks useful, as the corresponding α-attributes

have a large difference in scores. On the other hand, a comparison between w_4 and w_3 would be of little use. Suppose, then, that we are given that w_6 is less than 2.5 times w_1. We can examine the effect of this constraint by considering the two linear programs which maximize and minimize $\Sigma\Delta_i$, respectively. Considering the minimization first, without any constraints the solution is to let α_2=1. This solution satisfies the constraint that $w_6 < 2.5w_1$ (which is that $\alpha_1 < 1.5 \sum_{i=2}^{6} \alpha_i/_i$) because α_1=0, so that the constraint has no effect on this problem. In other words, this constraint says nothing about the possibility that option 2 might be better than option 3. In the maximization problem, however, the free maximum occurs when α_1=1, and this does <u>not</u> satisfy the constraint, so that the constraint does have an effect on this problem. An opposite constraint between w_6 and w_1, will similarly affect the minimization problem, so that, for example, if we are given that w_6 is also greater than 1.5 times w_1, then α_1 must be allocated a certain weight, because we have that

$$\alpha_1 > .5 \sum_{i=2}^{6} \alpha_i/_i \,.$$

Thus the pair of assessments provides one useful constraint for each problem.

Recalling the analysis of Section 2, if a constraint of the form

$$\sum_{i=j}^{n} \alpha_i/_i < \lambda \sum_{i=k}^{n} \alpha_i/_i$$

is met, then a basic solution will consist of two non-zero α-attributes, one between j and k-1, and one after k-1. Furthermore, the maximum will be positive if there are x and y such that $(\lambda-1)x\Delta_x + y\Delta_y$ is positive, and $j<x<k$, $y\geq k$. Now, by taking the last column of Table 4, and multiplying all numbers between j and k-1 by $(\lambda-1)$, we can very easily see if a positive maximum exists. In this case, we have j=1, k=2, and λ=2.5 in the maximization case, and j=1, k=2, and λ=1.5 in the minimization case. The new figures are shown in Table 5, with the third column being the maximization, and the fourth the minimization. Looking at the third

TABLE 5

α-attribute	w-attribute	option 3 over 2 (maximization)	option 2 over 3 (minimization)
α_1	w_6	4.8	1.6
α_2	w_1	-4.4	-4.4
α_3	w_2	-4.2	-4.2
α_4	w_5	-4.4	-4.4
α_5	w_4	7.5	7.5
α_6	w_3	8.4	8.4

column, the maximum occurs when α_1 and α_6 are non-zero. Moreover, all basic solutions involving two non-zero α-attributes are positive. It can be shown that this means that it is impossible to find comparisons which will make the maximum negative, unless a reappraisal is made of this first constraint.

Inspection of the fourth column reveals that the minimum occurs when the two α-attributes are α_1 and either α_2 or α_4. In this case, however, there are some basic solutions with positive values, namely the two pairs α_1, α_5 and α_1, α_6. Let us, therefore, consider this column further, that is we are looking to see if option 2 can dominate option 3.* We can see that in order for the objective function to be negative, we must take one of α_2, α_3, and α_4 as the attribute to complement α_1 in the basic solution, so that α_5 and α_6 must be zero. This suggests that a useful next assessment will be of the form $w_1 < \lambda w_4$, which will force a positive value on one of α_5 and α_6. Suppose that in this case, λ is given to be 3. Applying the analysis of Section 2 again, we can adapt the fourth column of Table 5 to include this constraint, as in Table 6. Here we must choose one α-attribute above the top line (i.e., α_1), one between the two lines, and one below the bottom line.

*N.B. If it is shown that one option can never dominate the other, with a given set of constraints, it does not necessarily mean that domination will occur the other way. To see this, consider Table 5 with the last two scores in each column being -4 each, instead of 7.5 and 8.4.

TABLE 6

α-attribute	w-attribute	option 2 over 3
α_1	w_6	4.8
α_2	w_1	-8.8
α_3	w_2	-8.4
α_4	w_5	-8.8
α_5	w_4	7.5
α_6	w_3	8.4

It is simple to see that the minimum occurs when α_1, α_2 or α_4, and α_5 are non-zero, and that the minimum value is 4.8-8.8+7.5=3.5, which is positive. That the minimum is positive means that all feasible solutions are positive, so that option 2 dominates option 3. The analysis of this problem need go no further, because enough information has been provided to produce a best option, namely option 2.

Of course, one would not expect such a simple analysis to succeed in general, but we can use the example to illustrate one or two further points. Firstly, it is possible to anticipate the possibility of an assessment providing a dominated option. For example, suppose we had obtained the first assessment in the example, so that Table 5 had been produced. The next assessment will be of the form $w_1 < \lambda w_4$. This in turn will mean that the minimum will be when α_1, α_2 or α_4, and α_5 are non-zero, and the minimum value will be:

$$1.6\lambda - 4.4(\lambda-1) + 7.5 = 11.9 - 2.8\lambda$$

This is negative if $\lambda > \frac{11.9}{2.8}$, i.e., if λ is greater than about 4. It is often likely that an analyst's intuitions of the problem will suggest whether λ will be in the region of this figure, and so whether domination can be expected or not.

Now, suppose that a further assessment was necessary. If this were between w_4 and w_3, and a multiplier λ were obtained, all the values above that of the α_5 row would be multiplied by λ. Thus, until we know this λ, the real relation between the 8.4 of the last row, and the values in the higher rows, is unknown. However, once there is a set of inequalities that link the first α-attribute and the last, further assessments will not continue to multiply up the value of the first α-attribute. We will call such a set of assessments a bounded set of assessments, because it places upper bounds on all the values for any further assessments.

Next, consider an assessment between two of the weights in the middle band, i.e., between two of w_1, w_2, and w_5. Suppose that $w_1<2.5w_4$, and that the next useful assessment were that $w_1<1.5w_2$. If this constraint is met, it effectively means that the implicit ratio between w_2 and w_4 is 2.5/1.5 = 1.67. The new α-value for α_2 is thus $(1.5-1)1.67i\Delta_i$, and for α_3 and α_4 are $(1.67-1)i\Delta_i$. The sum of these two multipliers is just (2.5-1), so that the sum of the two new values is just a weighted average of the values prior to the assessment. Note, however, that none of the values apart from these three will change, and this will always be the case once the set of assessments becomes bounded.

Finally, observe that the assessments that have so far been considered have been simple to analyze, because the constraints and multipliers have been specially chosen. Matters would become more complicated if, having made the first two assessments, an assessment between, say, w_6 and w_3 were made. In the first place, such an assessment might either be redundant, or else make another constraint redundant. In the second place, it requires much more effort to ascertain whether a greater than or less than assessment is required. Such redundancy is avoided by careful selection of assessments. Alternatively, it should be possible for a computer to be able to take a given set of assessments and eliminate any redundancies, and present the kind of table that the user can easily examine.

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